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The control of a wheeled mechanical system☆

V.I. Matyukhin

Moscow, Russia

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Abstract

Non-holonomic systems with rolling or wheeled systems are investigated. The investigation is restricted to kinematic models and the dynamics of the drive mechanism of the system are taken into account. A control law is constructed which stabilizes the motion of a wheeled system along a specified trajectory (a plane smooth curve). For the basic variables of the system, the property of stabilizability is substantiated in the large.

Investigations of wheeled systems (WS) include the analysis of theoretical problems (the stability of the motion of WS, planning of the motion, controllability etc.[1–9] and an analysis of applied problems of the control of wheeled systems (the stabilization of an automobile on a highway, automatic parking, ensurance of the motion of a robot in a medium with obstacles, etc.[10–12]).

The study of wheeled systems (WS) is usually restricted to simplied models, that is, to kinematic models which describe the motion of transport vehicles of different kinds fairly well.[1] Kinematic models only contain a subsystem of mechanical constraints, but difficulties arise even in this case. It has been found that smooth steady-state control laws do not allow one to stabilize the motions of a non-holonomic system in the general case.[2] The requirement of smoothness is important since only smooth control laws can be implemented when account is taken of the controlling drive mechanisms of a system. For example, the requirement of exponential stability is also important since, in practice, it is only in this case that perturbations of different kind are permitted. As a result of these facts, methods for constructing continuous, transient control laws are being developed.[9]

The important special control problem in which it is required to stabilize the motion of a WS along a specified trajectory (that is, along the manifold of the state space of the wheeled system[3]) is solved below. The need to stabilize the trajectory of a WS frequently arises in practice, such as in construction, soil processing[11,12] etc.

This control problem for the WS is investigated under natural assumptions. The trajectory of the WS is specified in the form of a planar smooth curve. Only a single control is used in the system, that is, control of the front axle of the WS. The dynamics of the axle drive mechanism are taken into account in a general form. The main result is related to the construction of a control law which stabilizes the motion of the WS. The property of stabilizability with respect to the basic variables (the position and orientation of the WS) is substantiated in the large.

A description of the dynamics of a WS is given in Section 1, the problem considered in this paper, that is, the construction of a control law for the WS, is then formulated (Section 2) and it is solved in a simplified linear formulation.

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E-mail address: matyukhin@ipu.rssi.ru.

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In Section 4, the control problem for the WS is formulated under the assumption that the specified trajectory of the motion is a straight line. It is shown that the motion of the WS will essentially be stabilized in the case of any initial deviations. This fact also holds in the general case when the specified trajectory of the motion is not necessarily a straight line (Section 5). The dynamics of the drive mechanism of the WS are discussed in a general way in Section 6. Finally, results of the modelling of the WS dynamics are presented in Section 7.

1. A Wheeled mechanical system as an object of control

The general scheme of the wheeled system (WS) being studied is shown in Fig. 1. The WS consists of a frame and a driving rear axle as well as a steering front axle. The state of the frame is characterized by the angle a and also by the coordinates x and y of a certain point of it p. The modulus of its velocity is denoted by $\mathbf{V}$. The state of the front axle is characterized by the controlled angle b.

Taking the notation used into account, we shall describe the motion of the WS by the system of equations

$$\dot{x} = v\cos a, \quad \dot{y} = v\sin\alpha, \quad \dot{a} = v\,\mathrm{tg}\,b/L, \quad \dot{b} = F(b, u, t) \tag{1.1}$$

The first three equations specify the translational and angular motion of the WS. The last equation describes the dynamics of the drive mechanism of the controlled front axle where u is the control and $\mathbf{V}$, $L = \text{const} > 0$. The initial aim of the control is that the point p of the WS with coordinates x and y should move along the specified curve S (Fig. 1).

The first two relations of (1.1) describe the mechanical constraint of the wheeled system in Fig. 1. The significance of the constraint lies in the fact that the rear wheels do not slip in a direction along the axes of the wheels. The relation $\dot{x}_1 = w\cos(a+b)$, $\dot{y}_1 = w\sin(a+b)$, where x_1 and y_1 are the coordinates of the point p_1 and w is the modulus of its velocity, describes the analogous assumption for the front wheels. The third equation of (1.1) is constructed on this basis.

The first three equations of system (1.1) constitute the well-known kinematic model of a WS.[1] The model only reflects those dynamical properties of a real WS which are associated with the kinematics. Generalizations of the model have therefore been made.[1] For example, in certain papers, the inertial properties of the WS are taken into account and the masses and moments of inertia of the frame and the wheel axles of the system are allowed for.[10] In such a wide formulation, it is possible to consider, for instance, features of the interaction between the wheels of the WS and the surface of the road.[8–10] Note also that the system with rolling being studied is a classical object of study within the framework of the analytical mechanics of non-holonomic systems.[13]

Note that, as a rule, the dynamics of the drive mechanisms of a WS are not taken into account. Instead of this, for example, the variable b in Eq. (1.1) has been considered as a control parameter, let us say, from the class of continuous or smooth functions of time. Below, the dynamics of the drive mechanism of the WS is taken into account within the framework of the last equation of system (1.1). Hydraulic drives are conventionally used as the drive mechanism of the front axle of a practical WS.[14] The above equation describes their most common properties. In particular, account is taken of the fact that the description of the dynamics of the drive mechanism of a WS usually contains undetermined parameters. The function F in system (1.1) is therefore not assumed to be known; only some of its properties (Section 4) are assumed to be known.

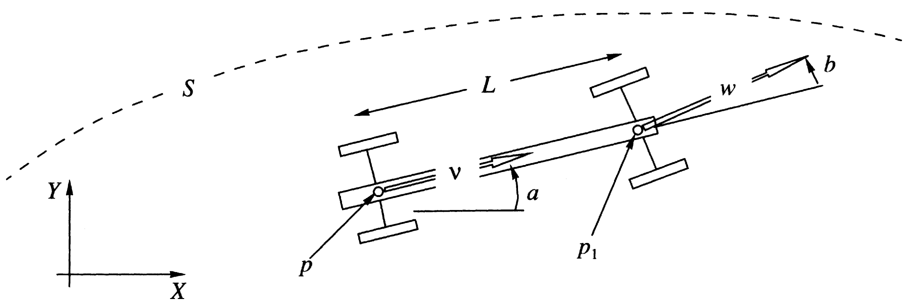

Fig. 1.

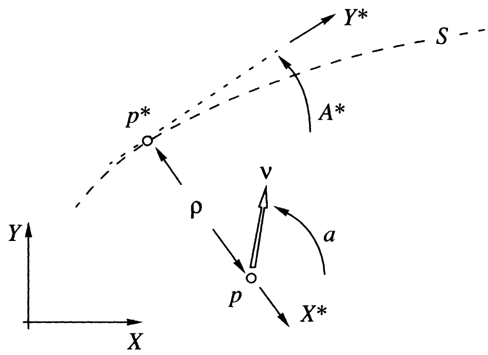

Fig. 2.

2. Formulation of the problem

We will formalize the initial target $p \in S$ in the problem of the control of the wheeled system (1.1) with the relations

$$\rho = 0 \tag{2.1}$$

$$\rho^2 \overset{\text{def}}{=} \min_{(x^1, y^1) \in S} \langle (x - x^1)^2 + (y - y^1)^2 \rangle \tag{2.2}$$

Here x and y are the coordinates of the point p and x^*, y^* is the solution of problem (2.2) for minimum. The point p^* with these coordinates lies on the curve S and is closest to the point p (Figs. 1 and 2). According to relation (2.1), the aim of the WS control will be achieved if $x = x^*$ and $y = y^*$.

Conditions are considered (Section 5) for which, when they are satisfied, the solution of problem (2.2) is unique. In particular, it is assumed that the smooth plane curve S has a bounded curvature, and points p near the curve are investigated.[5] It is also assumed that the exact description of the curve S is known (and can be used in the control law). We note that, in practice, a description of the curve S is available in the onboard computer of the WS. It can be automatically created and stored in the memory, for example. At this stage, satellite position detectors are usually employed and they are subsequently used for the operational control of a WS in automatic mode.[15]

Together with the aim of the control (2.1), other aims of a WS control, such as maintaining the requirements concerning the speed of motion along the trajectory S, are also considered. In applied problems, together with condition (2.1), a number of conditions of a constraining nature (for example, the limits within which the angle b of the front axle of the WS can be varied, the lateral acceleration of the WS and, also, other dynamic characteristics[10]) have to be taken into account. In problems involving the automatic parking of a WS, the trajectory S is not specified in advance.[1] The trajectory is constructed (planned) for specified initial and final positions of the WS. In the general case, account is taken of the orientation of the frame of the WS and its velocity, as well as surrounding obstacles, including moving obstacles.

The aim of this paper is to construct a control law $u = U(x, y, a, b)$ which ensures the stability of the motion (2.1) of the closed system (1.1). Note that relation (2.1) specifies the set of points (x, y) in the X,Y plane which belong to the curve S, that is, it defines a certain manifold of the state space x, y, a, b of the system. The subsequent discussion therefore concerns the stability of the manifold (2.1) of system (1.1).[3,16,17] It is assumed that the state x, y, a, b of the system is known as well as the constants $\mathbf{V}$ and L, that is, these parameters can be used in the control law.

3. Solution of the linear control problem

For simplicity, we will first solve the problem of the control of a wheeled system in the linear formulation. To do this, we introduce a system of the form

$$\dot{x} = v(\pi/2 - a), \quad \dot{y} = v, \quad \dot{a} = vb/L, \quad \dot{b} = -q_1 b + q_2 u; \quad q_1 = \text{const} \geq 0 \tag{3.1}$$

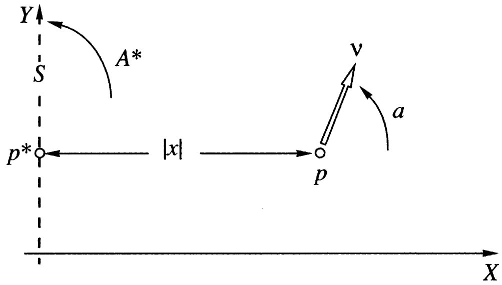

Fig. 3.

This system can be considered as a linear approximation of the initial system (1.1) in the neighbourhood of its motion

$$x = 0, \quad y = y(0) + vt, \quad a = \pi/2, \quad b = 0 \tag{3.2}$$

We shall also assume that the specified trajectory S is a straight line (Fig. 3). In this case, the description (2.2) of the distance to the curve S takes the simplest form $\rho = |x|$. The aim of the system control (3.1) is therefore to ensure the equality $x = 0$. Taking the notation which has been introduced into account, system (3.1) can be written in the form

$$\dddot{x} = -q_1\ddot{x} - q_2v^2u/L, \quad \dot{y} = v$$

When the linear control

$$u = q_3\ddot{x} + q_4\dot{x} + q_5x$$

is used, system (3.1) takes the form

$$\dddot{x} + q_6\ddot{x} + q_7\dot{x} + q_8x = 0, \quad \dot{y} = v$$

3.1. Existence theorem

Suppose the equation of motion of the WS has the form (3.1) and the specified trajectory of the motion S is the straight line $x = 0$ (Fig. 3). Then, a linear control exists which ensures the stability of a motion of the WS of the form $x = 0$ along the straight line S.

The theorem is only presented in order to demonstrate the existence of a solution of the control problem in question. It is applicable to a motion of the WS of the form (1.1) in a certain small domain of the motion (3.2). This domain corresponds to the following conditions: the WS moves close to the specified trajectory of the motion, that is, close to the straight line S, the angle of orientation a of the system is almost the same as the orientation $A^* = \pi/2$ of the trajectory, and the angle b of the front axle is fairly small (Fig. 3).

The linear control introduced above ensures the stability of the motion of the WS in the case of small initial deviations from the target of the control and the transition process can be long in the general case. Moreover, system (3.1) contains many simplifications which have been presented above. The solution of the WS control problem is investigated below in the general case.

4. Solution of the control problem in a non-linear formulation

We shall now consider the equations of motion of the WS in the initial non-linear form (1.1). The main simplification is the assumption that the specified trajectory of the motion S is a straight line, as in the preceding section. The aim is to demonstrate the existence of a solution of the WS control problem in the non-linear case.

As has already been mentioned above, it is not assumed that the function F in the last equation of (1.1) is known. However, it is known that the function F is bounded and satisfies the conditions

$$F(b, h, t) \geq H, \quad F(b, -h, t) \leq -H, \quad \forall b, \forall t, |u| \leq h; \quad H, h = \text{const} > 0 \tag{4.1}$$

Hence, the angular velocity $\dot{b}$ of the controlled front axle of the WS is bounded and the control u can, in practice, only change the sign of the velocity $\dot{b}$ in accordance with conditions (4.1). Conditions (4.1) reflect the most important and essentially requisite properties of any real control device. Note that, instead of (4.1), it would be possible to consider a more complex description which reflects the possible special features of a practical WS drive mechanism. For example, conditions (4.1) can only hold in a certain domain of the state space $\{b\}$ of the drive mechanism, they may not be symmetric (4.1), etc.

A control law of the form

$$u = -h\,\mathrm{sign}(b - b_z), \quad \dot{a}_z \overset{\mathrm{def}}{=} v\,\mathrm{tg}(b_z)/L - \varphi(a - a_z), \quad \cos a_z \overset{\mathrm{def}}{=} f(x) \tag{4.2}$$

is constructed for the control of the WS. The quantity a_z is defined by the last relation, and the second relation defines the quantity b_z, where f and φ are certain functions. When account is taken of Eq. (1.1), the construction (4.2) which has been introduced enables us to construct the relations

$$\dot{e} = \varphi(e) + v(\mathrm{tg}b - \mathrm{tg}b_z)/L, \quad e = a - a_z, \quad \dot{x} = vf(x) + v(\cos a - \cos a_z) \tag{4.3}$$

The idea is to achieve a sliding process of the form $b = b_z$ in the system using the discontinuous control (4.2). In this case, the first relation takes the form

$$\dot{e} = \varphi(e); \quad e = a - a_z \tag{4.4}$$

The function $\varphi(\mathbf{e})$ is chosen such that the motion $e = 0$ of system (4.4) is exponentially stable, that is, $a = a_z$. It then follows from the third equation of (4.3) that

$$\dot{x} \approx vf(x) \tag{4.5}$$

The function $f(x)$ is analogous to the function $\varphi(x)$ and the stability of the motion $x = 0$ of the WS is established from this.

Theorem 1. *Suppose conditions (4.1) hold in the case of system (1.1), the arbitrary positive numbers a^0, b^0, x^0, y^0 ($0 < b^0 < \pi/2$) are given and the following inequalities hold*

$$|a(0)| \le a^0, \quad |b(0)| \le b^0, \quad |x(0)| \le x^0, \quad |y(0)| \le y^0 \tag{4.6}$$

Then a control of the form of (4.2) exists which ensures the stability of the motion $x = 0$ of the system along the straight line S.

The proof of the theorem is based on the following assertion, the justification for which is presented in Appendix 1.

Auxiliary theorem 1. Under the condition of Theorem 1, a sliding process

$$|b - b_z| \le \pi, \quad 0 \le t \le t^1, \quad b = b_z, \quad t \ge t^1 \tag{4.7}$$

arises in system (1.1), (4.2) and the following limits hold

$$|b_z| \le \bar{b}_z, \quad |b| \le \bar{b}, \quad t \ge 0; \quad \bar{b}_z, \bar{b} = \mathrm{const} < \pi/2 \tag{4.8}$$

In the sliding process (4.7), it follows from relations (4.3) that

$$\dot{e} = \varphi(e), \quad t \ge t^1$$

Hence it follows that

$$|e(t)| \le \Lambda|e(t^1)|\exp(-\lambda(t - t^1)), \quad t \ge t^1$$

The existence of the constants λ, $\Lambda > 0$ follows from the assumed stability of the system $\dot{e} = \varphi(e)$ whence

$$d|x|/dt \le -v|f| + v\Lambda|e(t^1)|\exp(-\lambda(t - t^1)), \quad t \ge t^1 \tag{4.9}$$

Here, relations (4.3), written in the form

$$d|x|/dt \le -v|f| + R, \quad d|e|/dt \le -|\varphi| + R_1$$
$$R = v|\cos a_z - \cos a| = v|\sin(\tilde{a}_z)e| \le v|e|, \quad R_1 = v|\mathrm{tg}\, b - \mathrm{tg}\, b_z|/L, \quad e = a - a_z \tag{4.10}$$

have been taken into account.

The number $e(t^1)$ in inequality (4.9) exists. In fact, the relations for a perturbation R_1

$$R_1 < \bar{R}_1, \quad t \ge 0, \quad R_1 = 0, \quad t \ge t^1; \quad \bar{R}_1 = v(\mathrm{tg}\,\bar{b} + \mathrm{tg}\,\bar{b}_z)/L \tag{4.11}$$

follow from relations (4.10) in the process (4.7) when account is taken of limits (4.8). On the basis of relations (4.10), the limits

$$d|e|/dt \le \bar{R}_1, \quad |e(t)| \le |e(0)| + t^1\bar{R}_1, \quad t \le t^1$$

are established from this.

Account is then taken of the fact that

$$e(0) = a(0) - a_z(0), \quad \cos a_z(0) = -f(r(0))$$

and the following limit is constructed

$$|e| \le \bar{e}, \quad 0 \le t \le t^1; \quad \bar{e} = a^0 + \pi + t^1\bar{R}_1 \tag{4.12}$$

The existence of the number $e(t^1)$ in inequality (4.9), that is, the exponential stability of the motion $x = 0$ of system (1.1), (4.2) follows from this. Theorem 1 therefore follows from assertions (4.7) and (4.8) of the auxiliary theorem 1.

Theorem 1 confirms the existence of a solution of the WS control problem in the general non-linear formulation when the dynamics of the drive mechanism are taken into account. It is important that there are no constraints on the initial values $x(0)$, $y(0)$ and $a(0)$ for the basic variables of the state of the WS. At the same time, the initial value of $b(0)$ can lie within a relatively wide range of $(-\pi/2, \pi/2)$. This constraint is natural from a formal point of view since, here, Eq. (4.1) for the motion of the WS do not have singularities. The above-mentioned constraint is also not restrictive from a practical point of view since it is relatively easy to achieve the corresponding position $b(0)$ of the front axle of the WS. Hence, the control (4.2) ensures the stable motion of the system for any initial values over the wide range of (4.6). Note that, in the general case, control law (4.2) depends on the numbers a^0, b^0, x^0, y^0. Hence, in Theorem 1, the topic of discussion concerns the stabilizability of the system intrinsically in the large. We also note that continuous laws (Section 6) can be used instead of discontinuous control law (4.2). The law (4.2) is the limiting case of these laws and is considered in order to simplify the discussion.

5. Control taking account of the curvature of the trajectory of motion

Unlike in the preceding section, here we will consider the general case when the specified trajectory of the motion S is not necessarily a straight line.

As the specified trajectory of the motion of the WS, we will consider plane curves S, which are defined parametrically

$$x = \Phi_x(s), \quad y = \Phi_y(s), \quad (d\Phi_x/ds)^2 + (d\Phi_y/ds)^2 \ne 0$$

The parameter s specifies the x and y coordinates of points of the curve S in the system $\{X, Y\}$ (Figs. 1 and 2) and has the meaning of a length of an arc of the curve S, $\mathbf{A}(s)$ is the angle of the tangent to the curve, $A^* = A^*(s^*)$, and s^* is the value of s for the point $(x^*, y^*) \in S$ which is closest to the point (x, y). The main assumption is that the derivatives of the function $A(s)$ with respect to the parameter s are bounded:

$$|A'| \le \bar{A}', \quad |A''| \le \bar{A}''; \quad \bar{A}', \bar{A}'' = \mathrm{const} \ge 0, \quad A' = dA(s)/ds \tag{5.1}$$

Hence, the curvature A' of the curve S and its derivative turn out to be bounded.

Taking the notation into account, the system control can be formalized in the form

$$r = 0 \tag{5.2}$$

Here, r is the abscissa of the point p in the system $\{X^*, Y^*\}$ in Fig. 2. It is obvious that $\rho = |r|$ and the initial aim of the control (2.1) $\rho = 0$ follows from relation (5.2).

The conditions under which the quantity r is uniquely defined are considered. Suppose problem (2.2) for the minimum is solved at a certain instant of time up to the instant $t = 0$ when the motion of the WS commences. If this problem has several solutions, then one of them is chosen (arbitrarily, for example). We will denote the value of s for the chosen solution (x^*, y^*) by s^{*0}, that is, $s^*(0-0) = s^{*0}$. Next, when $t \geq 0$, problem (2.2) reduces to the following local problem

$$s^*(t) = \arg\min_{|s - s^{*1}| \leq d} \langle (x(t) - \Phi_x(s))^2 + (y(t) - \Phi_y(s))^2 \rangle, \quad s^{*1}(t) = s^*(t-0) \tag{5.3}$$

When account is taken of assumption (5.1), this problem has a unique solution, since the number $d > 0$ is fairly small, $d < 1/\bar{A}'$. Consequently, when $t \geq 0$, the system of coordinates $\{X^*, Y^*\}$ and, also, the quantity r in (5.2) are uniquely defined.

The quantity r which has been introduced satisfies the following assertion: the relations

$$\dot{s}^*(1 + rA'^*) = v\cos(a - A^*), \quad \dot{r} = -v\sin(a - A^*), \quad A^* = A(s^*) \tag{5.4}$$

hold for system (1.1) and a curve S, corresponding to conditions (5.1).

The idea behind the proof of these relations is associated with the analysis of the vector equality

$$\mathbf{P}^* + r\mathbf{n} = \mathbf{P}$$

where $\mathbf{P}$ and $\mathbf{P}^*$ are the radius vectors of the points p and p^* in the system $\{X, Y\}$ (Fig. 2), and $\mathbf{n}$ and $\boldsymbol{\tau}$ are the unit vectors of the system of coordinates $\{X^*, Y^*\}$. By virtue of system (1.1), the derivative $\dot{\mathbf{P}}^* + \dot{r}\mathbf{n} + r\dot{A} * \boldsymbol{\tau} = \dot{\mathbf{P}}$ is projected onto the unit vectors $\mathbf{n}$ and $\boldsymbol{\tau}$, and relations (5.4) follow from this.

We shall consider the first relation of (5.4) as a differential equation for the quantity $s^*(t)$. It is solvable for the higher derivative $\dot{s}^*$, if $1 + rA'(s^*) \neq 0$, which is satisfied since the variable r will change, for example, in the domain $|r| < 1/\bar{A}$ (see the proof of Theorem 2). In this case, the solution $s^*(t)$ of the first equation of (5.4) will also give a smooth solution of problem (5.3). Hence, system (1.1), (5.4) is actually being investigated in this paper in which the first two equations of subsystem (1.1) play a secondary role. Here, it is possible to confine ourselves just to using initial values $x(0)$ and $y(0)$ for the variables of these equations in the solution of problem (5.3) for a minimum.

A control (4.2) of the form

$$u = -h\,\mathrm{sign}(b - b_z), \quad \upsilon\,\mathrm{tg}(b_z)/L \overset{\text{def}}{=} \dot{a}_z + \varphi(a - a_z), \quad \sin(a_z - A^*) \overset{\text{def}}{=} -f(r) \tag{5.5}$$

is used to control the WS. The control law explicitly contains the curvature $A^* = A(s^*)$ of the specified curve S, unlike the control (4.2). Relations (5.5) which have been introduced enable us to construct the equation $\dot{r} \approx \mathbf{V}f(r)$, which is analogous to Eq. (4.5). In other words, Eq. (5.5) enables us to stabilize the motion $r = 0$ of the wheeled system.

Theorem 2. *Suppose conditions (4.1) hold in the case of system (1.1), the arbitrary numbers a^0, b^0, x^0, y^0 $(0 < b^0 < \pi/2)$ are given and that the following inequalities hold*

$$|a(0)| \leq a^0, \quad |b(0)| \leq b^0, \quad |x(0)| \leq x^0, \quad |y(0)| \leq y^0 \tag{5.6}$$

Then, under assumption (5.1), the numbers $\bar{A}'$ and $\bar{A}''$ exist and a control of the form (5.5) also exists which ensures the stability of the motion $r = 0$ of the system along the curve S.

The proof of the theorem follows the proof of the analogous Theorem 1. In fact, the basic difference from Theorem 1 is associated with the fact that the curve S is not necessarily a straight line. Nevertheless, the assertions of Theorem 1 are found to hold essentially in this case also. For example, relations (4.7) and (4.8) hold in the case of system (1.1), (5.5) which is being considered.

Auxiliary theorem 2. Under the conditions of Theorem 2, the sliding process (4.7) arises in system (1.1), (5.5) and the estimates (4.8) hold.

A proof of this theorem is presented in Appendix 2.

The inequality:

$$d|r|/dt \le -\nu|f| + \nu\Lambda\left|e(t^1)\right|\exp(-\lambda(t-t^1)),\quad t \ge t^1 \tag{5.7}$$

analogous to inequality (4.9), is established on the basis of the assertion which has been presented.

The existence of the number $e(t^1)$ is based on the estimate

$$\begin{aligned}&|e| \le \bar{e},\quad 0 \le t \le t^1\\ &\bar{e} = a^0 + A^0 + \pi + \arcsin(\bar{f}) + t^1\bar{R}_1;\quad A^0 = \max_{|x(0)|<x^0,\,|y(0)|<y^0} |A(s^*(0))|\end{aligned} \tag{5.8}$$

Here, account has been taken of the fact that the quantity A^* is contained in control law (5.5), unlike in the law (4.2). Hence,

$$e(0) = a(0) - a_z(0),\quad \sin(a_z(0) - A(s^*(0))) = -f(r(0))$$

and estimate (5.8) is not the same as estimate (4.12), which is unimportant.

The meaning of Theorem 2 lies in the fact that a solution of the problem exists in the general non-linear formulation when the specified trajectory of the motion of the WS has the general form of a smooth plane curve. As in Theorem 1, there is essentially no constraint on the initial values ($x(0)$, $y(0)$, $a(0)$, $b(0)$). Condition (5.1) for the curvature of the trajectory of the motion S is a natural condition. This is related to the fact that the aim of the control must be realizable, that is, a certain (unperturbed) motion of the WS must exist which corresponds to a given curve S, and this is only possible when S is sufficiently smooth.

6. Continuous control laws

A law for the control of a WS must not necessarily have a discontinuous form of feedback of the type of (5.5). A continuous control law can be constructed, for example, by following the schemes for constructing the quantities b_z and a_z in relations (5.5). To do this, the equation $\dot{b} = F(b, u, t)$ for the drive mechanism of the WS is written in the form

$$\dot{q} = -\dot{b}_z + F(q + b_z, u, t);\quad q = b - b_z \tag{6.1}$$

Suppose the continuous time function $u_z = u_z(t)$ satisfies Eq. (6.1) of the form

$$\psi(q) = -\dot{b}_z + F(q + b_z, u_z, t) \tag{6.2}$$

In this case, Eq. (6.1) takes the form

$$\dot{q} = \psi(q) + F(q + b_z, u, t) - F(q + b_z, u_z, t) \tag{6.3}$$

The control $u = u_z(t)$ converts system (6.3) into the system $\dot{q} = \psi(q)$. The trivial solution of this system is exponentially stable (the function ψ is so constructed). Consequently, $b \to b_z$ and the motion of the system will be stable.

Note that, in the arguments presented, it is necessary to assume that the function F is known, unlike in the assumption in Section 4. Furthermore, instead of assumption (4.1), it is necessary to introduce the assumption that the function F allows of a continuous solution $u_z = u_z(t)$ of problem (6.2). Note also that, within the framework of scheme (6.1)–(6.3), it is possible to take account of the dynamics of the drive mechanism of the WS in the general form $\dot{b} = F(b, u, t)$, $\dot{u} = \Omega(b, u, U, t)$. Here, the quantity u is regarded as a variable of the state $\{b, u\}$ of the drive mechanism. The new control U must ensure that $u \to u$. Finally, we note that the method being considered for constructing the control of a WS in the form of a relation of the type (6.3) is not general but it does enable us

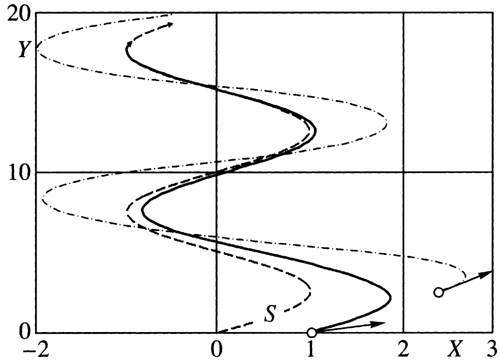

Fig. 4.

to take account of the special features of the dynamics of the WS. In particular, the method enables us to guarantee that a condition of the type $|f| < 1$ is satisfied, when the relation $sin(a_z - A^*) = -f(r)$ in definition (5.5) makes sense.

7. Modelling of a wheeled system (WS)

The aim of the modelling is to illustrate the practical direction of the theoretical results presented above. A WS with control (5.5) was investigated using a computer. The following values of the main dynamical parameters of the system were chosen: the length $L = 3$ m, the velocity $v = 2$ m/s and the speed of response of the front axle drive mechanism $|\dot{b}| \leq 1/c$. The specified trajectory of the motion of the system was chosen in the form of a sine curve with an amplitude of 1 m and a period of 10 m. The radius of curvature of this curve reached a value of 3 m. The parameters which have been presented are important and their values are typical of tractors used for agricultural purposes.[11,12]

The trajectory of the motion of the WS is shown in Fig. 4 by the solid curve with its origin at the point $x(0) = 1$, $y(0) = 0$. The specified curve S is the sine curve depicted by the dashed curve. The trajectory of the motion of the front axle of the WS is also shown in Fig. 4 (the dot-dash curve). The main result of the modelling is as follows. A fairly short transient time is ensured in the closed system. In fact, the deviation of the position of the WS from the curve S by 1 m after 7 s is practically compensated and becomes less than 0.01 m and subsequently decreases. The steady deviation from the specified trajectory of the motion is solely determined by the accuracy of the computerized integration process. This fact holds if the initial deviations satisfy the wide constraints $|a(0)| \leq 0.3$, $|b(0)| \leq 1$.

8. Appendix 1. Proof of auxiliary Theorem 1

By virtue of system (1.1), (4.2) being considered, the Lyapunov function $G = |b - b_z|$ has the form

$$\dot{G} = [\dot{b} - \dot{b}_z]\operatorname{sign}(b - b_z) = [F(b, -h\operatorname{sign}(b - b_z), t) - \dot{b}_z]\operatorname{sign}(b - b_z) \quad (8.1)$$

If $b \neq b_z$, then when account is taken of property (4.1), we obtain the following limit

$$\dot{G} \leq -H + |\dot{b}_z|$$

If the estimate

$$|\dot{b}_z| < H/2 \quad (8.2)$$

holds (see Lemma 1), the function $G(t)$ satisfies the differential inequality

$$\dot{G} < -H/2 \quad (8.3)$$

The solutions of this inequality fall to zero and the expressions[16–20]

$$G \le G(0) - Ht/2, \text{ when } 0 \le t < t_1, \quad G = 0, \text{ when } t \ge t_1; \quad t_1 = 2G(0)/H \tag{8.4}$$

therefore hold in the general case. Relation (4.7) follows from this.

Hence, the sliding process (4.7) follows from inequality (8.2).

Lemma 1. *Condition (8.2) holds when $t=0$ and, when $t>0$, it follows from inequality (8.3). The estimates (4.8) also hold.*

The proof of the lemma is presented below.

It follows from Lemma 1 that inequality (8.3) holds. In fact, inequality (8.3) is true when $t=0$, since, when $t=0$, inequality (8.2) is true according to Lemma 1. Inequality (8.3) is also true when $t>0$. This fact is proved by contradiction.

Hence, the theorem follows from Lemma 1.

Proof of Lemma 1. The expression

$$\dot{b}_z v/(L\cos^2 b_z) = \ddot{a}_z + (\dot{a} - \dot{a}_z)\varphi'$$

follows from the definition (4.2) for b_z. Condition (8.2) therefore follows from the inequality

$$[|\ddot{a}_z(t)| + (|\dot{a}| + |\dot{a}_z|)|\varphi'|]L/v < H/2, \quad t \ge 0 \tag{8.5}$$

The idea behind the proof of inequality (8.5) is associated with an analysis of constraints of the form

$$|\ddot{a}_z| \le \bar{\ddot{a}}_z, \quad |\dot{a}| \le \bar{\dot{a}}, \quad |\dot{a}_z| \le \bar{\dot{a}}_z, \quad t \ge 0; \quad \bar{\ddot{a}}_z, \bar{\dot{a}}, \bar{\dot{a}}_z = \text{const} \tag{8.6}$$

Account is also taken of the fact that the functions $\varphi(x)$ and $f(x)$ and their derivatives can be bounded by arbitrary positive constants $\bar{\varphi}, \ldots, \bar{f}''$

$$|\varphi| \le \bar{\varphi}, \quad |\varphi'| \le \bar{\varphi}', \quad |f| \le \bar{f} < 1, \quad |f'| \le \bar{f}', \quad |f''| \le \bar{f}'' \tag{8.7}$$

Inequality (8.5) holds if constraints (8.6) hold and the constants $\bar{\ddot{a}}_z$ and $\bar{\varphi}'$ are sufficiently small.

To prove inequalities (8.6), we specify the constant $\bar{\dot{a}}_z$ by the relations

$$L\bar{\dot{a}}_z/v < \mathrm{tg}(\bar{b}_z)/2, \quad \bar{b}_z = (\pi/2 - b^0)/4 < \pi/2 \tag{8.8}$$

where the condition $0 < b^0 < \pi/2$ of the theorem is taken into account.

Assertion 1.1. The third limit of (8.6) holds, where

$$\bar{\dot{a}}_z = \bar{f}' v/\sqrt{1 - \bar{f}^2} \tag{8.9}$$

This limit follows from the equalities

$$-\dot{a}_z \sin a_z = f'\dot{x} = f' v \cos a \tag{8.10}$$

which are constructed taken account of definition (4.2) and the equations of motion of the system being considered. The number $\bar{\dot{a}}_z$ satisfies inequality (8.8) since the numbers $\bar{f}'$ and $\bar{f}$ in assumption (8.7) are chosen to be sufficiently small.

The following limits are constructed on the basis of Assertion 1.1.

Limit (4.8) for the quantity b_z holds. In fact, the inequalities

$$|\mathrm{tg}\, b_z| \le L(|\dot{a}_z| + |\varphi|)/v \le L(\bar{\dot{a}}_z + \bar{\varphi})/v \le \mathrm{tg}(\bar{b}_z)/2 + L\bar{\varphi}/v \tag{8.11}$$

follow from definition (4.2) when account is taken of limit (8.9). The required limit follows from this. Here, it is necessary to take account of condition (8.8) as well as the inequality $L\bar{\varphi}/v < \mathrm{tg}(\bar{b}_z)/2$. This inequality is satisfied since the number $\bar{\varphi}$, $|\varphi| \le \bar{\varphi}$ is chosen to be sufficiently small, in accordance with condition (8.7).

Limit (4.8) for the quantity b holds, where

$$\bar{b} = b^0 + 2\bar{b}_z = \pi/4 + b^0/2 < \pi/2 \tag{8.12}$$

In fact, when $t = 0$, the limit follows from the condition $|b(0)| \leq b^0 < \pi/2$ of the theorem. When $t > 0$, assumption (8.3) of the lemma is true. Hence, relations (4.7) of the type

$$|b - b_z| \leq |b(0) - b_z(0)|, \quad 0 \leq t \leq t^1, \quad b = b_z, \quad t > t^1$$

hold for b. Consequently, the limit

$$|b| \leq |b_z| + |b(0) - b_z(0)| \leq |b_z| + b^0 + |b_z(0)|, \quad t > 0$$

holds and the required limit (4.8) follows from this.

Assertion 1.2. The second limit of (8.6) holds, where $\bar{a} = \mathrm{tg}(\bar{b})\mathbf{V}/L$, which follows from the equations of the system being considered and the limits (4.8).

Assertion 1.3. The first limit of (8.6) holds, where

$$\bar{\ddot{a}}_z = \langle\bar{\dot{a}}_z^2 + \bar{f}''v^2 + \bar{f}'v\bar{\dot{a}}\rangle/\sqrt{1 - \bar{f}^2} \tag{8.13}$$

Actually,

$$-\ddot{a}_z\sin a_z - \dot{a}_z^2\cos a_z = f''v^2\cos^2 a - f'v\dot{a}\sin a$$

follows from relation (8.10). When account is taken of the second and third limits of (8.6), expression (8.13) follows from this. The number $\bar{\ddot{a}}_z$ (8.13) will be small which is required for the substantiation of inequality (8.5), since the constants $\bar{\varphi}, \ldots, \bar{f}''$ in (8.7) are fairly small and the constant $\bar{\dot{a}}_z$, introduced by equality (8.9), is small. Assertion 1.3 is proved.

Lemma 1, as well as the auxiliary Theorem 1 follow from Assertions 1.1–1.3.

9. Appendix 2. Proof of auxiliary theorem 2

The proof of this theorem follows the proof of the auxiliary theorem 1 which is analogous to it. Substantial differences only arise in the proof of Assertions 1.1–1.3 of Lemma 1. In particular, the following assertion, which is analogous to Assertion 1.1 holds.

Assertion 2.1. The third limit of (8.6) holds, where

$$\bar{\dot{a}}_z = \bar{A}'2v + \bar{f}'v/\sqrt{1 - \bar{f}^2} \tag{9.1}$$

unlike the analogous expression (8.9).

Actually, the equality

$$(\dot{a}_z - \dot{A}^*)\cos(a_z - A^*) = -f'\dot{r}$$

follows from definition (5.5).

The inequality

$$|\dot{a}_z| \leq \bar{A}'|\dot{s}^*| + \bar{f}'v/\sqrt{1 - \bar{f}^2}$$

is constructed on this basis, where the relations

$$\dot{A}^* = A'^*\dot{s}^*, \quad \cos^2(a_z - A^*) = 1 - f^2$$

are taken into account. If the limit

$$|\dot{s}^*| \le 2\nu, \quad t \ge 0 \tag{9.2}$$

holds, equality (9.1) follows from this.

Inequality (9.2) is constructed on the basis of the first assertion of (5.4), and the inequality

$$|\dot{s}^*| \le \nu/|1 + rA'^*| \le \nu/(1 - |r|\bar{A}')$$

follows from this. Hence, inequality (9.2) is established, if the number $\bar{A}'$ is so small that $\bar{r}\bar{A}' < 1/2$ and the limit

$$|r| < \bar{r}, \quad t \ge 0 \tag{9.3}$$

holds.

Inequality (9.3) is constructed on the basis of limit (5.7), and the relation

$$\frac{d|r|}{dt} \le -\nu|f| + \begin{cases} \nu\bar{e} & \text{when} \quad 0 \le t \le t^1 \\ \nu\Lambda\bar{e}\exp(-\lambda(t - t^1)) & \text{when} \quad t > t^1 \end{cases}$$

is established from this, taking account of limit (5.8).

Limit (9.3) follows from this, where, for example,

$$\bar{r} = r_0 + \nu\bar{e}(t^1 + \Lambda/\lambda), \quad r_0 = \max_{|x_0| < x^0,\, |y_0| < y^0} \sqrt{\langle x_0 - \Phi_x(s^*(0))\rangle^2 + \langle y_0 - \Phi_y(s^*(0))\rangle^2}$$

Note that limit (5.7) is constructed on the basis of limits (4.8). Inequality (9.3) therefore holds if limits (4.8) hold.

Limits (4.8) hold and are established in the same way as in Lemma 1. The difference lies on the fact that limit (4.8) for b_z is constructed on the basis of inequality (8.11), where equality (9.1) being proved is taken into account. In other words, this equality follows from limit (4.8) for b_z and vice versa. It can be shown that equality (9.1) is true when $t = 0$. It can be established by contradiction that it is also true when $t > 0$. Assertion 2.1 is proved.

The other assertions of Lemma 1 are proved in a similar way. The auxiliary Theorem 2 is proved.

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